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Depth Imaging Partnership

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SPACE-TIME TREFFTZ-DG APPROACH FOR ELASTO-ACOUSTIC WAVE PROPAGATION

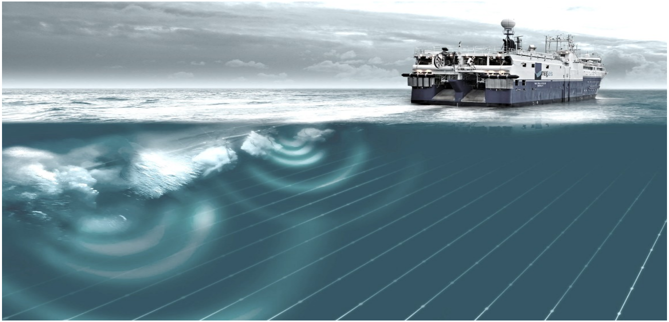
WCCM 2018 | 13th World Congress on Computational Mechanics

New York - USA | 2018

ABSTRACT

SEISMIC SURVEY

FIGURE 1: Ocean Bottom Seismic (OBS) data acquisition



BASIC NUMERICAL METHODS

TABLE 1: Generic properties of the most widely used numerical methods*

Numerical method	Complex geometries	High-order accuracy and *hp*-adaptivity	Explicit semi-discrete form	Conservation laws	Elliptic problems
FDM	🔴	●	●	●	●
FVM	●	🔴	●	●	⊙
FEM	●	●	🔴	⊙	●
DG-FEM	●	●	●	●	⊙

* J.S.Hesthaven T.Warburton. *Nodal DG methods. Algorithms, analysis, and applications.* 2007

DISCONTINUOUS GALERKIN METHODS

PROS AND CONS

Adapted to the complex geometries

High-order accuracy and hp-adaptivity

Explicit semi-dicrete form

Conservation laws

Higher number of degrees of freedom,

compared to the methods with continuous approximation

TREFFTZ METHOD

DEFINITION *

Given a region of an Euclidean space of some partitions of that region,

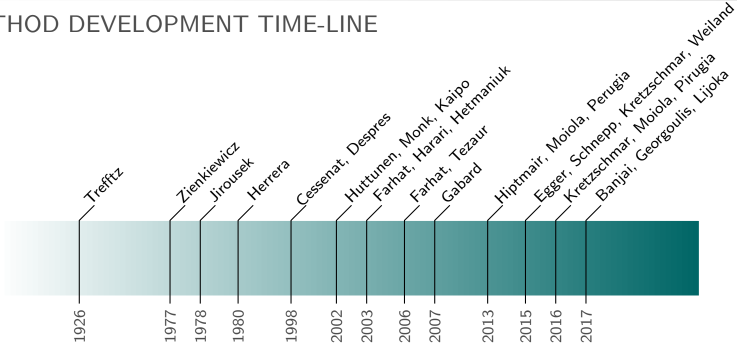
a **Trefftz method** is any procedure for solving BVP of PDE,

using solutions of that PDE or its adjoint, defined in its subregions.

* I.Herrera. *Trefftz method: a general theory.* 2000

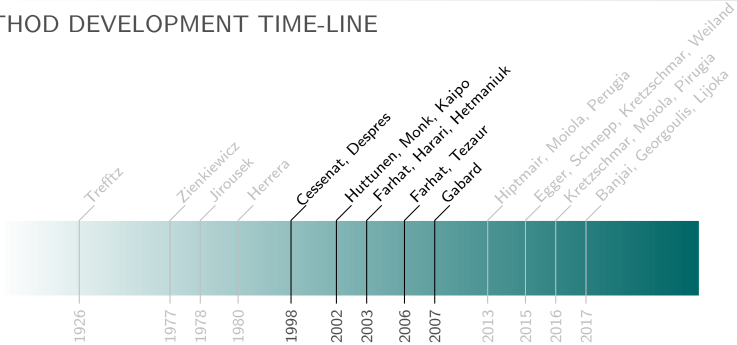
TREFFTZ METHOD

METHOD DEVELOPMENT TIME-LINE



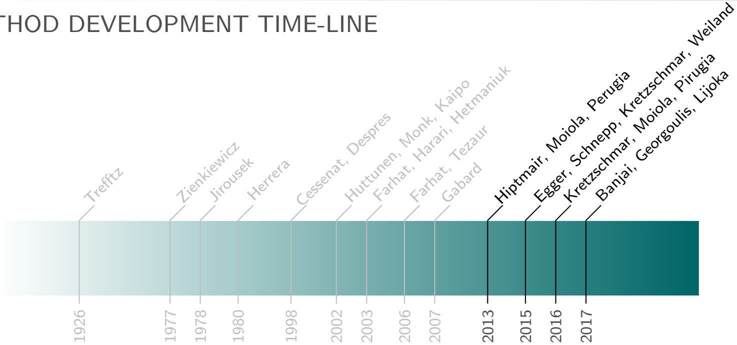
TREFFTZ METHOD

METHOD DEVELOPMENT TIME-LINE



TREFFTZ METHOD

METHOD DEVELOPMENT TIME-LINE



TREFFTZ METHOD

EXPECTED ADVANTAGES AND DRAWBACKS

- Higher order of convergence
- Flexibility in the choice of basis functions
- Low dispersion
- Incorporation of propagation directions in the discrete space
- Adaptivity and local space-time mesh refinement

TREFFTZ METHOD

EXPECTED ADVANTAGES AND DRAWBACKS

Higher order of convergence

Flexibility in the choice of basis functions

Low dispersion

Incorporation of propagation directions in the discrete space

Adaptivity and local space-time mesh refinement

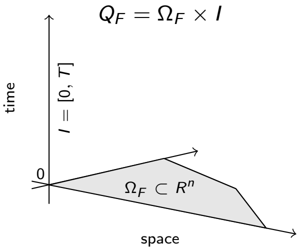
Sparse form of the global space-time matrix

MATHEMATICAL FORMULATION

ACOUSTIC SYSTEM

ACOUSTIC SYSTEM

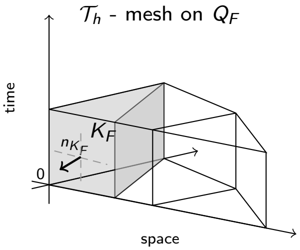
PROBLEM EQUATIONS



$$\begin{cases} \dfrac{1}{c_F^2 \rho_F} \dfrac{\partial p}{\partial t} + \operatorname{div} v_F = f & \text{in } Q_F \\ \rho_F \dfrac{\partial v_F}{\partial t} + \nabla p = 0 & \text{in } Q_F \\ v_F = v_{F\,0}, \; p = p_0 & \text{in } \Omega_F \times \{0\} \\ v_F = g_F^D & \text{in } \partial\Omega_F \times I \end{cases}$$

ACOUSTIC SYSTEM

MESHING



$\mathcal{F}_h = \cup_{K_F \in \mathcal{T}_h} \partial K_F$ - mesh skeleton

$K_F \in Q_F$ $(c_F, \rho_F \equiv const$ in $K_F)$

$v_F,\ p \in H^1(K_F)$

$\omega_F,\ q \in H^1(K_F)$

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT K_F

AS

(v_F, p)

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT K_F

AS
(v_F, p)

$\times$

test functions
(ω_F, q)

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT K_F

$$\underbrace{\boxed{\begin{array}{c}\text{AS}\\ (v_F,\, p)\end{array}} \times \boxed{\begin{array}{c}\text{test functions}\\ (\omega_F,\, q)\end{array}}}_{\text{SPACE-TIME INTEGRATION ON } K_F} =$$

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT K_F

VOLUME INTEGRATION TERM

AS (test functions) $\times$ (v_F, p)

AS (v_F, p) $\times$ test functions (ω_F, q) $=$

SPACE-TIME INTEGRATION ON K_F

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT K_F

VOLUME INTEGRATION TERM

AS (test functions) $\times$ (v_F, p)

AS (v_F, p) $\times$ test functions (ω_F, q) $=$ $+$

SPACE-TIME INTEGRATION ON K_F

SURFACE INTEGRATION TERM

(test functions) (v_F, p)

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT K_F

VOLUME INTEGRATION TERM

AS (test functions) $\times$ (v_F, p)

AS (v_F, p) $\times$ test functions (ω_F, q) $=$

$+$

SURFACE INTEGRATION TERM

(test functions) (v_F, p)

SPACE-TIME INTEGRATION ON K_F

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT K_F

VOLUME INTEGRATION TERM

test functions in T

AS (v_F, p) $\times$ test functions (ω_F, q) $=$

$+$

SURFACE INTEGRATION TERM

(test functions) (v_F, p)

SPACE-TIME INTEGRATION ON K_F

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT K_F

VOLUME INTEGRATION TERM

$$= 0$$

AS (v_F, p) $\times$ test functions (ω_F, q) $=$ VOLUME INTEGRATION TERM $+$ SURFACE INTEGRATION TERM

SPACE-TIME INTEGRATION ON K_F

SURFACE INTEGRATION TERM

(test functions) (v_F, p)

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION IN THE ELEMENT K_F

SURFACE INTEGRATION TERM

$$\underbrace{\boxed{\text{AS } (v_F, p)} \times \boxed{\text{test functions } (\omega_F, q)}}_{\text{SPACE-TIME INTEGRATION ON } K_F} = \boxed{\text{(test functions) } (v_F, p)}$$

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION

SUM OVER ALL K_F | | SUM OVER ALL ∂K_F

SURFACE INTEGRATION TERM

$$\underbrace{\overbrace{\boxed{\text{AS}\;(v_F,\,p)} \times \boxed{\text{test functions}\;(\omega_F,\,q)}}} = \boxed{(\text{test functions})\;(v_F,\,p)}$$

SPACE-TIME INTEGRATION ON K_F

ACOUSTIC SYSTEM

SPACE-TIME DISCRETIZATION

$$
-\sum_{K_F}\int_{K_F}\Big[p_h\big(\frac{1}{c_F^2\rho_F}\frac{\partial q}{\partial t}+\mathrm{div}\omega_F\big)+v_{F\,h}\cdot\big(\rho_F\frac{\partial \omega_F}{\partial t}+\nabla q\big)\Big]d\mathrm{v}
$$

$$
+\sum_{K_F}\int_{\partial K_F}\Big[\big(\frac{1}{c_F^2\rho_F}\hat{p}_h q+\rho_F\hat{v_F}_h\cdot\omega_F\big)n^t_{K_F}+\big(\hat{p}_h\omega_F+\hat{v_F}_h q\big)\cdot n^x_{K_F}\Big]d\mathrm{s}=\sum_{K_F}\int_{K_F} fq d\mathrm{v}
$$

ACOUSTIC SYSTEM

SPACE-TIME DISCRETIZATION

$$-\sum_{K_F}\int_{K_F}\Big[p_h\big(\frac{1}{c_F^2\rho_F}\frac{\partial q}{\partial t}+\mathrm{div}\omega_F\big)+v_{F\,h}\cdot\big(\rho_F\frac{\partial \omega_F}{\partial t}+\nabla q\big)\Big]d\mathrm{v}$$

$$+\sum_{K_F}\int_{\partial K_F}\Big[\big(\frac{1}{c_F^2\rho_F}\hat{p}_h q+\rho_F \hat{v_F}_h\cdot\omega_F\big)n^t_{K_F}+\big(\hat{p}_h\omega_F+\hat{v_F}_h q\big)\cdot n^x_{K_F}\Big]d\mathrm{s}=\sum_{K_F}\int_{K_F} fq d\mathrm{v}$$

ACOUSTIC SYSTEM

SPACE-TIME DISCRETIZATION

$$-\sum_{K_F}\int_{K_F}\Big[p_h\big(\frac{1}{c_F^2\rho_F}\frac{\partial q}{\partial t}+\mathrm{div}\omega_F\big)+v_{F\,h}\cdot\big(\rho_F\frac{\partial \omega_F}{\partial t}+\nabla q\big)\Big]dv$$

$$+\sum_{K_F}\int_{\partial K_F}\Big[\big(\frac{1}{c_F^2\rho_F}\hat{p}_h q+\rho_F \hat{v_F}_h\cdot\omega_F\big)n^t_{K_F}+\big(\hat{p}_h\omega_F+\hat{v_F}_h q\big)\cdot n^x_{K_F}\Big]ds=\sum_{K_F}\int_{K_F} fq\,dv$$

TREFFTZ SPACE

$$\mathbf{T}_F(\mathcal{T}_h)\equiv\Big\{(\omega_F,q)\subset H^1(\mathcal{T}_h)^2:\forall K_F\in\mathcal{T}_h,\ \rho_F\frac{\partial\omega_F}{\partial t}+\nabla q=\frac{1}{c_F^2\rho_F}\frac{\partial q}{\partial t}+\mathrm{div}\omega_F=0\Big\}$$

ACOUSTIC SYSTEM

SPACE-TIME DISCRETIZATION

$$-\sum_{K_F}\int_{K_F}\left[p_h\left(\frac{1}{c_F^2\rho_F}\frac{\partial q}{\partial t}+\mathrm{div}\omega_F\right)+v_{F\,h}\cdot\left(\rho_F\frac{\partial\omega_F}{\partial t}+\nabla q\right)\right]d\mathrm{v}$$

$$+\sum_{K_F}\int_{\partial K_F}\left[\left(\frac{1}{c_F^2\rho_F}\hat{p}_h q+\rho_F\hat{v_F}_h\cdot\omega_F\right)n_{K_F}^t+\left(\hat{p}_h\omega_F+\hat{v_F}_h q\right)\cdot n_{K_F}^x\right]d\mathrm{s}=\sum_{K_F}\int_{K_F} fq\,d\mathrm{v}$$

TREFFTZ SPACE

$$\mathbf{T}_F(\mathcal{T}_h)\equiv\left\{(\omega_F,q)\subset H^1(\mathcal{T}_h)^2:\forall K_F\in\mathcal{T}_h,\ \rho_F\frac{\partial\omega_F}{\partial t}+\nabla q=\frac{1}{c_F^2\rho_F}\frac{\partial q}{\partial t}+\mathrm{div}\omega_F=0\right\}$$

SPACE-TIME DISCRETIZATION IN $\mathbf{T}_F$

$$\sum_{K_F}\int_{\partial K_F}\left[\left(\frac{1}{c_F^2\rho_F}\hat{p}_h q+\rho_F\hat{v_F}_h\cdot\omega_F\right)n_{K_F}^t+\left(\hat{p}_h\omega_F+\hat{v_F}_h q\right)\cdot n_{K_F}^x\right]d\mathrm{s}=\sum_{K_F}\int_{K_F} fq\,d\mathrm{v}$$

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION

SUM OVER
ALL ∂K_F

SURFACE
INTEGRATION
TERM

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION

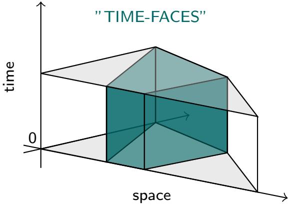
SUM THROUGH ALL ∂K_F

SURFACE INTEGRATION TERM

=

SUM THROUGH ALL "TIME-FACES"

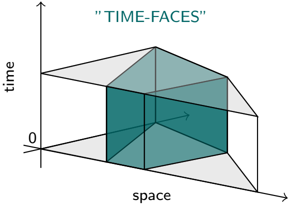
SURFACE INTEGRATION TERM

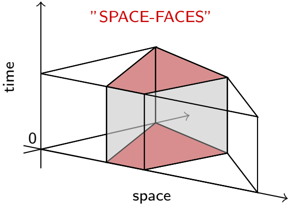


ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION

$$\underbrace{\boxed{\text{SURFACE INTEGRATION TERM}}}_{\text{SUM THROUGH ALL } \partial K_F} = \underbrace{\boxed{\text{SURFACE INTEGRATION TERM}}}_{\text{SUM THROUGH ALL "TIME-FACES"}} + \underbrace{\boxed{\text{SURFACE INTEGRATION TERM}}}_{\text{SUM THROUGH ALL "SPACE-FACES"}}$$

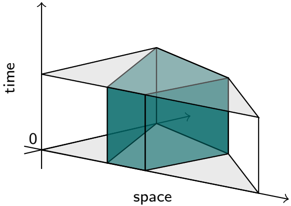




ACOUSTIC SYSTEM

MESH NOTATIONS

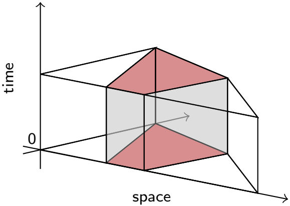
"TIME-FACES"



internal $\mathcal{F}_h^{I_F}$ $(x - fixed)$

boundary $\mathcal{F}_h^{D_F}$ $(\partial\Omega_F \times [0, T])$

"SPACE-FACES"



internal $\mathcal{F}_h^{\Omega_F}$ $(t - fixed)$

initial $\mathcal{F}_h^{0_F}$ $(\Omega_F \times \{0\})$

final $\mathcal{F}_h^{T_F}$ $(\Omega_F \times \{T\})$

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION

$$\sum_{\partial K_F} \int_{\partial K_F} \left[\left(\frac{1}{c_F^2 \rho_F} \hat{p}_h q + \rho_F \hat{v_F}_h \cdot \omega_F \right) n^t_{K_F} + \left(\hat{p}_h \omega_F + \hat{v_F}_h q \right) \cdot n^x_{K_F} \right] ds = \sum_{K_F} \int_{K_F} f q d\mathrm{v}$$

NUMERICAL FLUXES THROUGH THE ELEMENT FACES

$$\mathcal{F}_h^{I_F} \quad \begin{pmatrix} \hat{v_F}_h \\ \hat{p}_h \end{pmatrix} \equiv \begin{pmatrix} \{v_{F\,h}\} + \beta[\![p_h]\!]_x \\ \{p_h\} + \alpha[\![v_{F\,h}]\!]_x \end{pmatrix}$$

$$\mathcal{F}_h^{D_F} \quad \begin{pmatrix} \hat{v_F}_h \cdot n^x_{K_F} \\ \hat{p}_h n^x_{K_F} \end{pmatrix} \equiv \begin{pmatrix} \mathrm{g}_{D_F} \cdot n^x_{K_F} \\ p_h n^x_{K_F} + \alpha(v_{F\,h} - \mathrm{g}_{D_F}) \cdot n^x_{K_F} \end{pmatrix}$$

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION

$$\sum_{\partial K_F}\int_{\partial K_F}\left[\left(\frac{1}{c_F^2\rho_F}\hat{p}_h q+\rho_F \hat{v_F}_h\cdot\omega_F\right)n^t_{K_F}+\left(\hat{p}_h\omega_F+\hat{v_F}_h q\right)\cdot n^x_{K_F}\right]ds=\sum_{K_F}\int_{K_F} fq\,d\mathrm{v}$$

NUMERICAL FLUXES THROUGH THE ELEMENT FACES

$$\mathcal{F}_h^{I_F}\quad\begin{pmatrix}\hat{v_F}_h\\ \hat{p}_h\end{pmatrix}\equiv\begin{pmatrix}\{v_{F\,h}\}+\beta[\![p_h]\!]_x\\ \{p_h\}+\alpha[\![v_{F\,h}]\!]_x\end{pmatrix}$$

$$\mathcal{F}_h^{D_F}\quad\begin{pmatrix}\hat{v_F}_h\cdot n^x_{K_F}\\ \hat{p}_h n^x_{K_F}\end{pmatrix}\equiv\begin{pmatrix}\mathrm{g}_{D_F}\cdot n^x_{K_F}\\ p_h n^x_{K_F}+\alpha(v_{F\,h}-\mathrm{g}_{D_F})\cdot n^x_{K_F}\end{pmatrix}$$

$$\mathcal{F}_h^{\Omega_F}\quad\begin{pmatrix}\hat{v_F}_h\\ \hat{p}_h\end{pmatrix}\equiv\begin{pmatrix}v_{F\,h}^-\\ p_h^-\end{pmatrix}$$

$$\mathcal{F}_h^{T_F}\quad\begin{pmatrix}\hat{v_F}_h\\ \hat{p}_h\end{pmatrix}\equiv\begin{pmatrix}v_{F\,h}\\ p_h\end{pmatrix}$$

$$\mathcal{F}_h^{0_F}\quad\begin{pmatrix}\hat{v_F}_h\\ \hat{p}_h\end{pmatrix}\equiv\begin{pmatrix}v_{F0}\\ p_0\end{pmatrix}$$

ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION

Seek $(v_{F\,h}, p_h) \subset \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}_F(\mathcal{T}_h)$ s.t. for all $(\omega_F, q) \subset \mathbf{V}(\mathcal{T}_h)$ it holds:

$$\mathcal{A}_{TDG_F}((v_{F\,h}, p_h); (\omega_F, q)) = \ell_{TDG_F}(\omega_F, q)$$

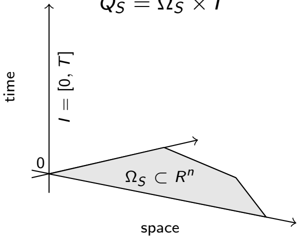
MATHEMATICAL FORMULATION

ELASTODYNAMIC SYSTEM

ELASTODYNAMIC SYSTEM

PROBLEM EQUATIONS

$$Q_S = \Omega_S \times I$$



$$\begin{cases} \mathbf{A}\dfrac{\partial \boldsymbol{\sigma}}{\partial t} - \boldsymbol{\varepsilon}(v_S) = 0 & \text{in } Q_S \\ \rho_S \dfrac{\partial v_S}{\partial t} - \operatorname{div}\boldsymbol{\sigma} = 0 & \text{in } Q_S \\ & \\ v_S = v_{S0}, \ \boldsymbol{\sigma} = \boldsymbol{\sigma}_0 & \text{in } \Omega_S \times \{0\} \\ \boldsymbol{\sigma} = \mathbf{g}_{D_S} & \text{in } \partial\Omega_S \times I \end{cases}$$

ELASTODYNAMIC SYSTEM

TREFFTZ-DG FORMULATION

$$\sum_{K_S} \int_{\partial K_S} \left[(\mathbf{A}\,\hat{\boldsymbol{\sigma}}_h : \boldsymbol{\xi} + \rho_S \hat{v_S}_h \cdot \omega_S) n^t_{K_S} - (\hat{v_S}_h \boldsymbol{\xi} + \hat{\boldsymbol{\sigma}}_h \omega_S) \cdot n^x_{K_S} \right] ds = 0$$

NUMERICAL FLUXES THROUGH THE ELEMENT FACES

$$\begin{array}{llll}
\mathcal{F}_h^{I_S} & \begin{pmatrix} \hat{v_S}_h \\ \hat{\boldsymbol{\sigma}}_h \end{pmatrix} & \equiv & \begin{pmatrix} \{v_{S\,h}\} - \delta[\![\boldsymbol{\sigma}_h]\!]_x \\ \{\boldsymbol{\sigma}_h\} - \gamma[\![v_{S\,h}]\!]_x \end{pmatrix} \\
\mathcal{F}_h^{D_S} & \begin{pmatrix} \hat{v_S}_h \cdot n^x_{K_S} \\ \hat{\boldsymbol{\sigma}}_h n^x_{K_S} \end{pmatrix} & \equiv & \begin{pmatrix} v_{S\,h} \cdot n^x_{K_S} - \delta(\boldsymbol{\sigma}_h - \mathbf{g}_{D_S}) n^x_{K_S} \\ \mathbf{g}_{D_S} n^x_{K_S} \end{pmatrix} \\
\mathcal{F}_h^{\Omega_S} & \begin{pmatrix} \hat{v_S}_h \\ \hat{\boldsymbol{\sigma}}_h \end{pmatrix} & \equiv & \begin{pmatrix} v_{S\,h}^{-} \\ \boldsymbol{\sigma}_h^{-} \end{pmatrix} \\
\mathcal{F}_h^{T_S} & \begin{pmatrix} \hat{v_S}_h \\ \hat{\boldsymbol{\sigma}}_h \end{pmatrix} & \equiv & \begin{pmatrix} v_{S\,h} \\ \boldsymbol{\sigma}_h \end{pmatrix} \\
\mathcal{F}_h^{0_S} & \begin{pmatrix} \hat{v_S}_h \\ \hat{\boldsymbol{\sigma}}_h \end{pmatrix} & \equiv & \begin{pmatrix} v_{S0} \\ \boldsymbol{\sigma}_0 \end{pmatrix}
\end{array}$$

ELASTODYNAMIC SYSTEM

TREFFTZ-DG FORMULATION

Seek $(v_{S\,h}, \boldsymbol{\sigma}_h) \subset \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}_S(\mathcal{T}_h)$, s.t. for all $K_S \in \mathcal{T}_h$, $(\omega_S, \boldsymbol{\xi}) \subset \mathbf{V}(\mathcal{T}_h)$ it holds:

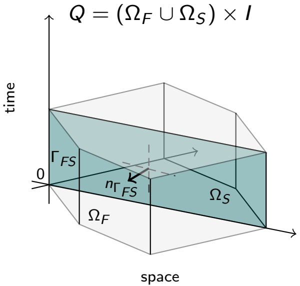
$$\mathcal{A}_{TDG_S}((v_{S\,h}, \boldsymbol{\sigma}_h); (\omega_S, \boldsymbol{\xi})) = \ell_{TDG_S}(\omega_S, \boldsymbol{\xi})$$

MATHEMATICAL FORMULATION

ELASTO-ACOUSTIC SYSTEM

ELASTO-ACOUSTIC SYSTEM

TRANSMISSION CONDITIONS



$$\begin{cases} v_F \cdot n_{\Gamma_{FS}} = v_S \cdot n_{\Gamma_{FS}} & \text{on } \Gamma_{FS} \\ \boldsymbol{\sigma} n_{\Gamma_{FS}} = -p n_{\Gamma_{FS}} & \text{on } \Gamma_{FS} \end{cases}$$

ELASTO-ACOUSTIC SYSTEM

TREFFTZ SPACE

$$\mathbf{T}(\mathcal{T}_h) \equiv \Big\{(\omega_F, q, \omega_S, \boldsymbol{\xi}) \subset H^1(\mathcal{T}_h)^4 : \ \rho_F \frac{\partial \omega_F}{\partial t} + \nabla q = \frac{1}{c_F^2 \rho_F} \frac{\partial q}{\partial t} + \mathrm{div}\omega_F = 0,$$

$$\rho_S \frac{\partial \omega_S}{\partial t} - \mathrm{div}\boldsymbol{\xi} = \frac{\partial \mathbf{A}\, \boldsymbol{\xi}}{\partial t} - \boldsymbol{\varepsilon}(\omega_S) = 0, \ \forall K_F, K_S \in \mathcal{T}_h \Big\}$$

ELASTO-ACOUSTIC SYSTEM

TREFFTZ SPACE

$$\mathbf{T}(\mathcal{T}_h) \equiv \Big\{(\omega_F, q, \omega_S, \boldsymbol{\xi}) \subset H^1(\mathcal{T}_h)^4 : \ \rho_F \frac{\partial \omega_F}{\partial t} + \nabla q = \frac{1}{c_F^2 \rho_F} \frac{\partial q}{\partial t} + \mathrm{div}\omega_F = 0,$$
$$\rho_S \frac{\partial \omega_S}{\partial t} - \mathrm{div}\boldsymbol{\xi} = \frac{\partial \mathbf{A}\,\boldsymbol{\xi}}{\partial t} - \boldsymbol{\varepsilon}(\omega_S) = 0, \ \forall K_F, K_S \in \mathcal{T}_h\Big\}$$

SPACE-TIME DISCRETIZATION IN T

$$\sum_{K_F} \int_{\partial K_F} \Big[\big(\frac{1}{c_F^2 \rho_F} \hat{p}_h q + \rho_F \hat{v_F}_h \cdot \omega_F\big) n^t_{K_F} + (\hat{p}_h \omega_F + \hat{v_F}_h q) \cdot n^x_{K_F} \Big] ds$$
$$+ \sum_{K_S} \int_{\partial K_S} \Big[(\mathbf{A}\,\hat{\boldsymbol{\sigma}}_h : \boldsymbol{\xi} + \rho_S \hat{v_S}_h \cdot \omega_s) n^t_{K_S} - (\hat{v_S}_h \boldsymbol{\xi} + \hat{\boldsymbol{\sigma}}_h \omega_S) \cdot n^x_{K_S} \Big] ds = \sum_{K_F} \int_{K_F} f q \, d\mathrm{v}$$

ELASTO-ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION

$$\sum_{K_F}\int_{\partial K_F}\Big[\Big(\frac{1}{c_F^2\rho_F}\hat{p}_h q+\rho_F\hat{v_F}_h\cdot\omega_F\Big)n^t_{K_F}+(\hat{p}_h\omega_F+\hat{v_F}_h q)\cdot n^x_{K_F}\Big]ds$$
$$+\sum_{K_S}\int_{\partial K_S}\Big[(\mathbf{A}\,\hat{\boldsymbol{\sigma}}_h:\boldsymbol{\xi}+\rho_S\hat{v_S}_h\cdot\omega_s)n^t_{K_S}-(\hat{v_S}_h\boldsymbol{\xi}+\hat{\boldsymbol{\sigma}}_h\omega_S)\cdot n^x_{K_S}\Big]ds=\sum_{K_F}\int_{K_F}fq\,dv$$

NUMERICAL FLUXES THROUGH THE FLUID-SOLID INTERFACE

$$\mathcal{F}_h^{\Gamma_{FS}}\qquad\begin{pmatrix}\hat{v_F}_h\cdot n^x_{K_F}\\ \hat{p}_h n^x_{K_F}\\ \hat{v_S}_h\cdot n^x_{K_S}\\ \hat{\boldsymbol{\sigma}}_h n^x_{K_S}\end{pmatrix}\equiv\begin{pmatrix}v_{S\,h}\cdot n^x_{K_F}\\ p_h n^x_{K_F}+\alpha(v_{F\,h}\cdot n^x_{K_F}-v_{S\,h}\cdot n^x_{K_F})\\ v_{S\,h}\cdot n^x_{K_S}-\delta(\boldsymbol{\sigma}_h n^x_{K_S}-p_h n^x_{K_S})\\ -p_h n^x_{K_S}\end{pmatrix}$$

ELASTO-ACOUSTIC SYSTEM

TREFFTZ-DG FORMULATION

Seek $(v_{F\,h}, p_h, v_{S\,h}, \boldsymbol{\sigma}_h) \subset \mathbf{V}(\mathcal{T}_h)$: $\forall K_F, K_S \in \mathcal{T}_h$, $(\omega_F, q, \omega_S, \boldsymbol{\xi}) \subset \mathbf{T}(\mathcal{T}_h)$ it holds:

$$\mathcal{A}_{TDG}((v_{F\,h}, p_h, v_{S\,h}, \boldsymbol{\sigma}_h); (\omega_F, q, \omega_S, \boldsymbol{\xi})) = \ell_{TDG}(\omega_F, q, \omega_S, \boldsymbol{\xi})$$

WELL-POSEDNESS

MESH DEPENDENT L^2 NORMS

ACOUSTIC TDG

ELASTODYNAMIC TDG

ELASTO-ACOUSTIC TDG

WELL-POSEDNESS

MESH DEPENDENT L^2 NORMS

ACOUSTIC TDG

ELASTODYNAMIC TDG

ELASTO-ACOUSTIC TDG

WELL-POSED*

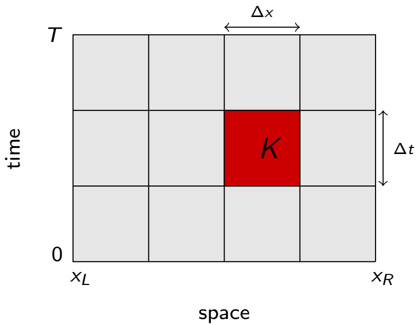
* H.Barucq H.Calandra J.Diaz E.Shishenina. *Space-time Trefftz-Discontinuous Galerkin approximation for elasto-acoustics.* [RR] 2017

IMPLEMENTATION OF THE ALGORITHM

MESH AND GLOBAL MATRIX INVERSION

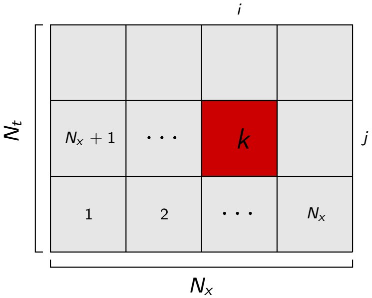
IMPLEMENTATION OF THE ALGORITHM

MESH AND ELEMENT NUMBERING



IMPLEMENTATION OF THE ALGORITHM

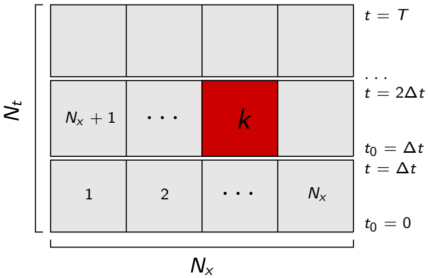
MESH AND ELEMENT NUMBERING



$$k = (j-1) \times N_x + i$$

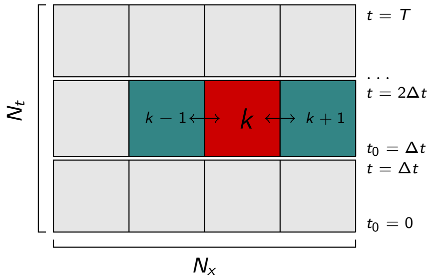
IMPLEMENTATION OF THE ALGORITHM

MESH AND ELEMENT NUMBERING



IMPLEMENTATION OF THE ALGORITHM

MESH AND ELEMENT NUMBERING



IMPLEMENTATION OF THE ALGORITHM

BILINEAR OPERATOR

$$\mathsf{A}_{TDG} \equiv \underbrace{\int_{\mathcal{F}_h^{\Omega}} + \int_{\mathcal{F}_h^{T}} + \int_{\mathcal{F}_h^{0}}}_{\mathcal{A}_{TDG}^{\Omega}} + \underbrace{\int_{\mathcal{F}_h^{I}} + \int_{\mathcal{F}_h^{D}}}_{\mathcal{A}_{TDG}^{I}}$$

IMPLEMENTATION OF THE ALGORITHM

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$$\mathrm{A}_{TDG} \equiv \underbrace{\int_{\mathcal{F}_h^\Omega} + \int_{\mathcal{F}_h^T} + \int_{\mathcal{F}_h^0}}_{\mathcal{A}_{TDG}^\Omega} + \underbrace{\int_{\mathcal{F}_h^I} + \int_{\mathcal{F}_h^D}}_{\mathcal{A}_{TDG}^I}$$

GLOBAL SPACE-TIME MATRIX

$$\mathbf{M} = \Delta_x M_\Omega + \Delta_t M_I$$

IMPLEMENTATION OF THE ALGORITHM

BILINEAR OPERATOR

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GLOBAL SPACE-TIME MATRIX

$$\mathbf{M} = \Delta_x M_{\Omega} + \Delta_t M_I$$

M_{Ω} - block-diagonal

IMPLEMENTATION OF THE ALGORITHM

BILINEAR OPERATOR

$$\mathrm{A}_{TDG} \equiv \underbrace{\int_{\mathcal{F}_h^{\Omega}} + \int_{\mathcal{F}_h^{T}} + \int_{\mathcal{F}_h^{0}}}_{\mathcal{A}_{TDG}^{\Omega}} + \underbrace{\int_{\mathcal{F}_h^{I}} + \int_{\mathcal{F}_h^{D}}}_{\mathcal{A}_{TDG}^{I}}$$

GLOBAL SPACE-TIME MATRIX

$$\mathbf{M} = \Delta_x M_{\Omega} + \Delta_t M_I$$

M_{Ω} - block-diagonal $\qquad$ M_I - sparse

IMPLEMENTATION OF THE ALGORITHM

GLOBAL MATRIX INVERSION

$$\mathbf{M}^{-1} \equiv \left[\Delta_x M_\Omega + \Delta_t M_I\right]^{-1}$$

IMPLEMENTATION OF THE ALGORITHM

GLOBAL MATRIX INVERSION

$$\mathbf{M}^{-1} \equiv \left[\Delta_x M_\Omega + \Delta_t M_I\right]^{-1}$$

$$=$$

$$\left[\,\mathbf{I} + \kappa\left(M_\Omega{}^{-1} M_I\right)\right]^{-1}\left[\Delta_x M_\Omega\right]^{-1}$$

IMPLEMENTATION OF THE ALGORITHM

GLOBAL MATRIX INVERSION

$$\mathbf{M}^{-1} \equiv \left[\Delta_x M_\Omega + \Delta_t M_I\right]^{-1}$$

$$=$$

$$\left[\mathbf{I} + \kappa\left(M_\Omega{}^{-1} M_I\right)\right]^{-1}\left[\Delta_x M_\Omega\right]^{-1}$$

TAYLOR EXPANSION (IN CONDITION OF SMALL ENOUGH κ)

$$\mathbf{M}^{-1} = \left[\sum_{n=0}^{\infty}(-1)^n \kappa^n \left(M_\Omega{}^{-1} M_I\right)^n\right]\left[\Delta_x M_\Omega\right]^{-1}$$

IMPLEMENTATION OF THE ALGORITHM

GLOBAL MATRIX INVERSION

$$\mathbf{M}^{-1} \equiv \left[\Delta_x M_\Omega + \Delta_t M_I\right]^{-1}$$

$$=$$

$$\left[\mathrm{I} + \kappa\left(M_\Omega{}^{-1} M_I\right)\right]^{-1}\left[\Delta_x M_\Omega\right]^{-1}$$

TAYLOR EXPANSION (IN CONDITION OF ENOUGH SMALL $\kappa \equiv \frac{\Delta_t^d}{\Delta_x^d}$)

$$\mathbf{M}^{-1} = \left[\sum_{n=0}^{\infty}(-1)^n \kappa^n \left(M_\Omega{}^{-1} M_I\right)^n\right]\left[\Delta_x M_\Omega\right]^{-1}$$

block-diagonal

IMPLEMENTATION OF THE ALGORITHM

GLOBAL MATRIX INVERSION

$$\mathbf{M}^{-1} \equiv \left[\Delta_x M_\Omega + \Delta_t M_I\right]^{-1}$$

$$=$$

$$\left[\mathrm{I} + \kappa \left({M_\Omega}^{-1} M_I\right)\right]^{-1} \left[\Delta_x M_\Omega\right]^{-1}$$

TAYLOR EXPANSION (IN CONDITION OF ENOUGH SMALL κ)

$$\mathbf{M}^{-1} = \left[\sum_{n=0}^{\infty} (-1)^n \kappa^n \left({M_\Omega}^{-1} M_I\right)^n\right] \left[\Delta_x M_\Omega\right]^{-1}$$

block-diagonal x sparse

IMPLEMENTATION OF THE ALGORITHM

APPROXIMATE INVERSION ($\kappa = 10^{-2}$)

n	$\Delta_x = 10^{-2}$	$\Delta_x = 2 \cdot 10^{-2}$	$\Delta_x = 5 \cdot 10^{-2}$	$\Delta_x = 10^{-1}$
3	1.4166E-05	4.3741E-05	2.8780E-04	2.5772E-03
4	3.1623E-07	1.2656E-06	5.3868E-05	1.2674E-03
5	2.8903E-07	9.1744E-07	4.1029E-05	1.3010E-03

FULL INVERSION ($\kappa = 10^{-2}$)

n	$\Delta_x = 10^{-2}$	$\Delta_x = 2 \cdot 10^{-2}$	$\Delta_x = 5 \cdot 10^{-2}$	$\Delta_x = 10^{-1}$
·	2.2540E-07	8.9583E-07	5.5811E-05	1.3004E-03

IMPLEMENTATION OF THE ALGORITHM

APPROXIMATE INVERSION ($\kappa = 10^{-2}$)

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FULL INVERSION ($\kappa = 10^{-2}$)

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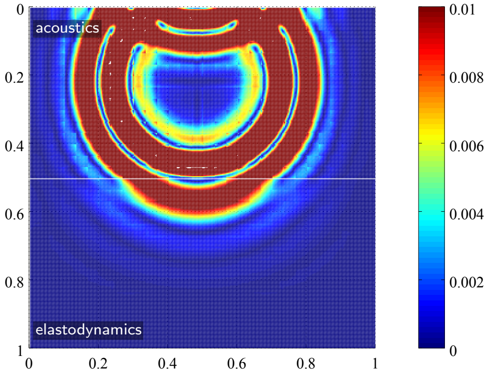
NUMERICAL RESULTS

2D + TIME SIMULATIONS

NUMERICAL RESULTS

2D ELASTO-ACOUSTIC SYSTEM

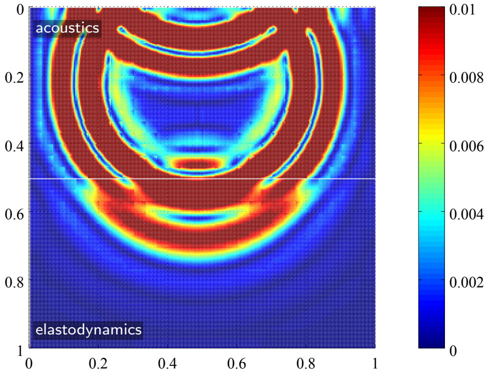
FIGURE 9: 2D Elasto-acoustic system.
Numerical velocity $v(x, 0.567)$. Dirichlet boundaries



NUMERICAL RESULTS

2D ELASTO-ACOUSTIC SYSTEM

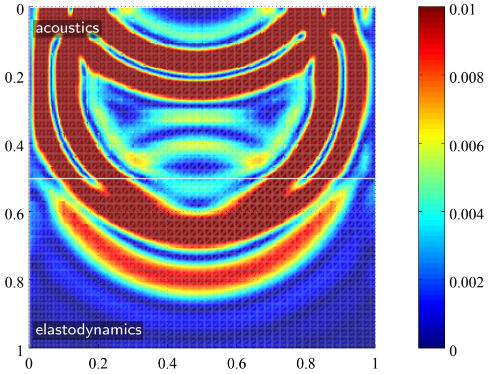
FIGURE 9: 2D Elasto-acoustic system.
Numerical velocity $v(x, 0.633)$. Dirichlet boundaries



NUMERICAL RESULTS

2D ELASTO-ACOUSTIC SYSTEM

FIGURE 9: 2D Elasto-acoustic system.
Numerical velocity $v(x, 0.700)$. Dirichlet boundaries



NUMERICAL RESULTS

2D ACOUSTIC AND ELASTODYNAMIC SYSTEMS

FIGURE 10: 2D Acoustic system.
Convergence of velocity v_F in function of cell size h

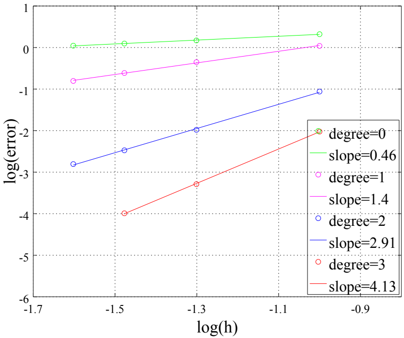
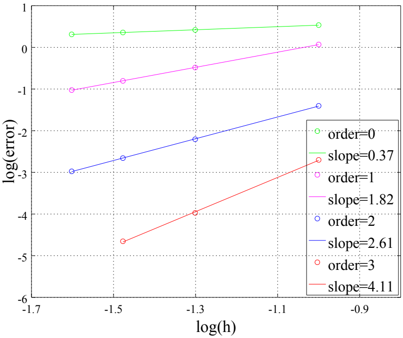


FIGURE 11: 2D Elastodynamic system.
Convergence of velocity v_S in function of cell size h



CONCLUSION

CONCLUSION

Implementation of the method for 2D acoustic, elastic, elasto-acoustic systems

Numerical validation of prototype Matlab code

Implementation of an algorithm of the approximate inversion

Development of 2D TDG acoustic propagator in the framework of Elasticus code

CONCLUSION

ON-GOING WORK AND PERSPECTIVES

Development of 2D TDG elastic propagator in the framework of Elasticus code; optimization

Exploration of space-time tent pitching meshes

Analysis of performance

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THANK YOU FOR ATTENTION!

IMPLEMENTATION OF THE ALGORITHM

POLYNOMIAL BASIS (ORDER 3)

p=0, ndof=2

$$\phi_1^v = 0 \qquad \phi_1^p = -c_F$$
$$\phi_2^v = 1 \qquad \phi_2^p = 0$$

p=1, ndof=4

$$\phi_3^v = x \qquad \phi_3^p = -c_F^2 t$$
$$\phi_4^v = c_F t \qquad \phi_4^p = -c_F x$$

p=2, ndof=6

$$\phi_5^v = -\frac{x^2}{2} - \frac{c_F^2 t^2}{2} \qquad \phi_5^p = c_F^2 xt$$
$$\phi_6^v = -c_F xt \qquad \phi_6^p = c_F\left(\frac{x^2}{2} + \frac{c_F^2 t^2}{2}\right)$$

p=3, ndof=8

$$\phi_7^v = -\frac{x^3}{6} - \frac{x c_F^2 t^2}{2} \qquad \phi_7^p = c_F\left(\frac{c_F^3 t^3}{6} + \frac{x^2 c_F t}{2}\right)$$
$$\phi_8^v = -\frac{c_F^3 t^3}{6} - \frac{x^2 c_F t}{2} \qquad \phi_8^p = c_F\left(\frac{x^3}{6} + \frac{x c_F^2 t^2}{2}\right)$$